TRANSFORMATIONS T OF CONJUGATE SYSTEMS OF CURVES ON A SURFACE*

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When a surface S is referred to a conjugate system of lines, its point coördinates are solutions of a partial differential equation of the Laplace type, called *the point equation* for the given conjugate system. Throughout this paper we consider surfaces referred to conjugate systems, and hence we will use the symbol S to denote either the surface or the parametric system upon it, as the case may be. When the developables of a linear congruence G meet S in the parametric conjugate system, we say that G and S are conjugate to one another. A second surface S_1 conjugate to G is said to be in the relation of a transformation T to S, or to be a T transform of S. Darboux† has shown that each solution ϕ of the adjoint equation of the point equation of S determines a congruence G conjugate to S, and that each solution of this point equation determines for any of these congruences G a conjugate surface S_1. Hence each pair of function ϕ and θ determines a transformation T, and every such transformation is so determined. It is the purpose of this paper to develop the theory of these transformations.

It is shown that if S_1 and S_2 are two T transforms of S, there exist ∞^2 surfaces S_{12} each of which is in the relation of transformations T with S_1 and S_2, and the determination of these surfaces requires only quadratures. We have thus proved the existence of a theorem of permutability of transformations T, which includes a similar theorem for the transformations K of conjugate systems with equal invariants‡ (see § 9), just as the latter embraces as a particular case the theorem established by Bianchi§ for transformations D_m of isothermic surfaces. In § 12 we extend the theorem of permutability so as to be concerned with eight surfaces.

When the function θ determining a transformation is a constant and the point coördinates are in the cartesian form, the corresponding tangent planes to S and S_1 are parallel, in which case we say that we have a parallel trans-

* Presented to the Society, Dec. 27, 1916.

† *Leçons*, vol. 2, pp. 225, 227.

‡ Cf. Eisenhart, These Transactions, vol. 15 (1914), pp. 404–8. Hereafter this memoir will be referred to as M_1.

§ Annali di Matematica, ser. 3, vol. 11 (1905), pp. 93–158.

formation. This result and the consideration of the relation between two transformations T determined by the same ϕ but different functions θ lead to results formerly found by the author* for certain types of transformations T and later by Jonas,† and enable us to put the equations of a general transformation T in another convenient form.

In § 9 we consider in particular the case where the point equation of S has equal invariants and its transforms possess the same property. The resulting transformations are the transformations‡ K previously studied by us in their relation to the transformations of Moutard of differential equations. As there shown, these transformations K include the transformations D_m of isothermic surfaces discovered by Darboux.§

Transformations T can be treated analytically also in terms of the tangential coördinates of the surface. This is done in § 10, and the relations between the two sets of equations are determined. In particular, the case where the tangential equation has equal invariants is studied, with the result that we are led to the transformations Ω previously discovered by the author.||

If x, y, z are the cartesian coördinates of a surface S and ω is any solution of the point equation of S, the surface $\bar{S}$ whose cartesian coördinates are $x/\omega, y/\omega, z/\omega$ is referred to a conjugate system. We say that $\bar{S}$ is a radial transform of S. Combinations of radial and T transformations are studied in § 13 in relation to the theorem of permutability of transformations T. In particular, it is shown that this theorem can be applied when a radial transformation is treated as a special type of transformation T.

1. Transformations T in homogeneous point coördinates

The necessary and sufficient condition that four functions, x, y, z, w, be the homogeneous point coördinates of a surface S, referred to a conjugate system of lines of parameters u and v, is that these functions satisfy an equation of the form

$$\frac{\partial^2 \theta}{\partial u \partial v} + a\frac{\partial \theta}{\partial u} + b\frac{\partial \theta}{\partial v} + c\theta = 0, \tag{1}$$

where a, b, c are functions of u and v. We refer to this equation as the point equation of the conjugate system.

If the developables of a rectilinear congruence meet S in the parametric

*Rendiconti di Palermo, vol. 39 (1915), pp. 153–176. Hereafter this memoir will be referred to as M_2.

†Sitzungsberichte der Berliner Mathematischen Gesellschaft, vol. 14 (1915), pp. 96–118.

‡ M_1, pp. 397–430.

§Annales de l'école normale superieure, ser. 3, vol. 16 (1899), pp. 491–508.

|| M_2.

curves, the congruence and the parametric system are said to be conjugate to one another. Darboux* has shown that when a solution ϕ of the adjoint equation of (1), namely

$$\frac{\partial^2 \phi}{\partial u \partial v} - a\frac{\partial \phi}{\partial u} - b\frac{\partial \phi}{\partial v} + \left(c - \frac{\partial a}{\partial u} - \frac{\partial b}{\partial v}\right)\phi = 0, \tag{2}$$

is known, a congruence G_1 conjugate to the parametric system is given by quadratures. In fact, the point coördinates, x_1', y_1', z_1', w_1'; x_1'', y_1'', z_1'', w_1'' of the focal points F_1' and F_1'' of the congruence are given by expressions of the form

$$\begin{aligned} x_1' &= \int \phi_1\left(\frac{\partial x}{\partial u} + bx\right)du + x\left(\frac{\partial \phi_1}{\partial v} - a\phi_1\right)dv, \\ x_1'' &= \int x\left(\frac{\partial \phi_1}{\partial u} - b\phi_1\right)du + \phi_1\left(\frac{\partial x}{\partial v} + ax\right)dv. \end{aligned} \tag{3}$$

Furthermore, each solution of (1) leads by quadratures to another conjugate system conjugate to the above congruence. For, if θ_1 is a solution of (1), the function σ_1 given by

$$\sigma_1 = \int \phi_1\left(\frac{\partial \theta_1}{\partial u} + b\theta_1\right)du + \theta_1\left(\frac{\partial \phi_1}{\partial v} - a\phi_1\right)dv, \tag{4}$$

is a solution of the point equation of the surface (F_1'), the locus of F_1'. Hence by the theorem of Levy† the surface S_1, whose coördinates are given by equations of the form

$$x_1 = x_1' - \frac{\sigma_1}{\dfrac{\partial \sigma_1}{\partial v}}\frac{\partial x_1'}{\partial v} = x_1 - \frac{\sigma_1}{\theta_1}x, \tag{5}$$

is conjugate to the congruence G_1 whose focal surfaces are (F_1') and (F_1''). We say that S_1 is obtained from S by a transformation T.

The equations of S_1 can be given another form, if we look upon the lines of the congruence G_1 as tangent to the focal surface (F_1'') also. Evidently the function τ_1, given by

$$\tau_1 = \int \theta_1\left(\frac{\partial \phi_1}{\partial u} - b\phi_1\right)du + \phi_1\left(\frac{\partial \theta_1}{\partial v} + a\theta_1\right)dv, \tag{6}$$

is a solution of the point equation of (F_1''). Hence the equations of S_1 can be given the form

$$x_1 = -x_1'' + \frac{\tau_1}{\theta_1}x. \tag{7}$$

* *Leçons*, vol. 2, p. 225.

† Journal de l'école polytechnique, cahier 56 (1886), p. 63; also Darboux, *Lecons*, vol. 2, p. 222.

From (4) and (6) we find the relation

$$\phi_1 \theta_1 = \sigma_1 + \tau_1. \tag{8}$$

From (5) or (7) we get by differentiation

$$\frac{\partial x_1}{\partial u} = \tau_1 \frac{\partial}{\partial u}\left(\frac{x}{\theta_1}\right), \quad \frac{\partial x_1}{\partial v} = -\sigma_1 \frac{\partial}{\partial v}\left(\frac{x}{\theta_1}\right). \tag{9}$$

2. The inverse of a transformation T

It is readily found from (9) that x_1, y_1, z_1, w_1 satisfy the equation

$$\frac{\partial^2 \theta_1'}{\partial u \partial v} - \frac{\sigma_1}{\tau_1}\left(a + \frac{\partial \log \theta_1}{\partial v}\right)\frac{\partial \theta_1'}{\partial u} - \frac{\tau_1}{\sigma_1}\left(b + \frac{\partial \log \theta_1}{\partial u}\right)\frac{\partial \theta_1'}{\partial v} = 0. \tag{10}$$

Since the relation between S and S_1 is reciprocal, there exist functions ϕ_1^{-1} and θ_1^{-1} by means of which S is the transform of S_1. We shall show that

$$\theta_1^{-1} = 1, \quad \phi_1^{-1} = \frac{\phi_1 \theta_1}{\sigma_1 \tau_1}. \tag{11}$$

We remark that as the congruence G_1 is the same, on the assumption that $\theta_1^{-1} = 1$, we must have analogously to (3)

$$\begin{aligned} \frac{\partial}{\partial u}(\rho x_1') &= \phi_1^{-1}\left[\frac{\partial x_1}{\partial u} - \frac{\tau_1}{\sigma_1}\left(b + \frac{\partial \log \theta_1}{\partial u}\right)x_1\right], \\ \frac{\partial}{\partial v}(\rho x_1') &= x_1\left[\frac{\partial \phi_1^{-1}}{\partial v} + \frac{\sigma_1}{\tau_1}\left(a + \frac{\partial \log \theta_1}{\partial v}\right)\phi_1^{-1}\right], \end{aligned} \tag{12}$$

where ρ is to be determined. If the value of x_1' from (5) be substituted in the first of these equations, the result is reducible to the form

$$A\frac{\partial x}{\partial u} + Bx = 0,$$

where A and B are determinate expressions. Since similar equations hold in y, z, and w, A and B must be equal to zero. From these equations we find that ρ is $1/\sigma$ and that ϕ_1^{-1} is of the form (11). It is readily shown that these values satisfy the second of (12).

From (4) and (6) we have

$$\begin{aligned} \frac{\partial^2 \sigma_1}{\partial u \partial v} &= \frac{1}{\theta_1 \phi_1}\frac{\partial \sigma_1}{\partial u}\frac{\partial \sigma_1}{\partial v} + \theta_1 \phi_1 k, \\ \frac{\partial^2 \tau_1}{\partial u \partial v} &= \frac{1}{\theta_1 \phi_1}\frac{\partial \tau_1}{\partial u}\frac{\partial \tau_1}{\partial v} + \theta_1 \phi_1 h, \end{aligned} \tag{13}$$

where h and k are the invariants of (1) and are given by

$$h = \frac{\partial a}{\partial u} + ab - c, \quad k = \frac{\partial b}{\partial v} + ab - c. \tag{14}$$

By means of (4) and (6) equation (10) may be given the form

$$\tag{15} \frac{\partial^2 \theta_1'}{\partial u \partial v} - \frac{\sigma_1}{\tau_1 \theta_1 \phi_1} \frac{\partial \tau_1}{\partial v} \frac{\partial \theta_1'}{\partial u} - \frac{\tau_1}{\sigma_1 \theta_1 \phi_1} \frac{\partial \sigma_1}{\partial u} \frac{\partial \theta_1'}{\partial v} = 0.$$

In consequence of (13) the adjoint of this equation is reducible to

$$\tag{16} \begin{aligned} &\frac{\partial^2 \phi_1'}{\partial u \partial v} + \frac{\sigma_1}{\tau_1 \theta_1 \phi_1} \frac{\partial \tau_1}{\partial v} \frac{\partial \phi_1'}{\partial u} + \frac{\tau_1}{\sigma_1 \theta_1 \phi_1} \frac{\partial \sigma_1}{\partial u} \frac{\partial \phi_1'}{\partial v} \\ &\qquad + \phi_1' \left[\frac{\sigma_1}{\tau_1} h + \frac{\tau_1}{\sigma_1} k + \frac{1}{\theta_1^2 \phi_1^2} \left(\frac{\partial \sigma_1}{\partial u} \frac{\partial \tau_1}{\partial v} + \frac{\partial \sigma_1}{\partial v} \frac{\partial \tau_1}{\partial u} \right) \right. \\ &\qquad \left. - \frac{1}{\theta_1 \phi_1} \left(\frac{\sigma_1}{\tau_1^2} \frac{\partial \tau_1}{\partial u} \frac{\partial \tau_1}{\partial v} + \frac{\tau_1}{\sigma_1^2} \frac{\partial \sigma_1}{\partial u} \frac{\partial \sigma_1}{\partial v} \right) \right] = 0. \end{aligned}$$

From (11) we find

$$\frac{\partial^2 \phi_1^{-1}}{\partial u \partial v} = \frac{1}{\tau_1^2} \frac{\partial \tau_1}{\partial u} \frac{\partial \tau_1}{\partial v} \frac{\tau_1 + 2\sigma_1}{\tau_1 \theta_1 \phi_1} + \frac{1}{\sigma_1^2} \frac{\partial \sigma_1}{\partial u} \frac{\partial \sigma_1}{\partial v} \frac{\sigma_1 + 2\tau_1}{\sigma_1 \theta_1 \phi_1} - \theta_1 \phi_1 \left(\frac{h}{\tau_1^2} + \frac{k}{\sigma_1^2} \right).$$

Making use of this result, we verify readily that ϕ_1^{-1} is a solution of (16).

If ϕ_1' is any solution of (16), then

$$\frac{\partial^2}{\partial u \partial v} \left(\frac{\phi_1'}{\phi_1^{-1}} \right) = \frac{\tau_1}{\theta_1 \phi_1 \sigma_1} \frac{\partial \sigma_1}{\partial v} \frac{\partial}{\partial u} \left(\frac{\phi_1'}{\phi_1^{-1}} \right) + \frac{\sigma_1}{\theta_1 \phi_1 \tau_1} \frac{\partial \tau_1}{\partial u} \frac{\partial}{\partial v} \left(\frac{\phi_1'}{\phi_1^{-1}} \right).$$

Hence if ϕ_1 and ϕ_2 are two solutions of (2), the equations

$$\tag{17} \frac{\partial}{\partial u} \left(\frac{\phi_{12}}{\phi_1^{-1}} \right) = \sigma_1 \frac{\partial}{\partial u} \left(\frac{\phi_2}{\phi_1} \right), \qquad \frac{\partial}{\partial v} \left(\frac{\phi_{12}}{\phi_1^{-1}} \right) = -\tau_1 \frac{\partial}{\partial v} \left(\frac{\phi_2}{\phi_1} \right)$$

are consistent, and the function ϕ_{12} so defined is a solution of (16).

3. Transformations T in cartesian coördinates. Parallel transformations T

We consider now the case when the point coördinates are non-homogeneous and rectangular. The point equation is of the form

$$\tag{18} \frac{\partial^2 \theta}{\partial u \partial v} + a \frac{\partial \theta}{\partial u} + b \frac{\partial \theta}{\partial v} = 0.$$

When we take $w = 1$, we have from the corresponding equation (9)

$$\tag{19} \frac{\partial w_1}{\partial u} = \tau_1 \frac{\partial}{\partial u} \left(\frac{1}{\theta_1} \right), \qquad \frac{\partial w_1}{\partial v} = -\sigma_1 \frac{\partial}{\partial v} \left(\frac{1}{\theta_1} \right).$$

Moreover, the cartesian coördinates x_1, y_1, z_1 of S_1 are given by equations of the form

$$\tag{20} \frac{\partial}{\partial u} (x_1 w_1) = \tau_1 \frac{\partial}{\partial u} \left(\frac{x}{\theta_1} \right), \qquad \frac{\partial}{\partial v} (x_1 w_1) = -\sigma_1 \frac{\partial}{\partial v} \left(\frac{x}{\theta_1} \right).$$

By means of (19) these equations are reducible to

$$\tag{21} \begin{aligned} \frac{\partial x_1}{\partial u} &= \frac{\tau_1}{w_1\,\theta_1^2}\left(\theta_1 \frac{\partial x}{\partial u} + (x_1 - x)\frac{\partial \theta_1}{\partial u}\right), \\ \frac{\partial x_1}{\partial v} &= -\frac{\sigma_1}{w_1\,\theta_1^2}\left(\theta_1 \frac{\partial x}{\partial v} + (x_1 - x)\frac{\partial \theta_1}{\partial v}\right). \end{aligned}$$

Now the point equation of S_1 is

$$\tag{22} \begin{aligned} \frac{\partial^2 \theta_1'}{\partial u \partial v} &+ \left[\frac{\partial \log w_1}{\partial v} - \left(a + \frac{\partial \log \theta_1}{\partial v}\right)\frac{\sigma_1}{\tau_1}\right]\frac{\partial \theta_1'}{\partial u} \\ &+ \left[\frac{\partial \log w_1}{\partial u} - \left(b + \frac{\partial \log \theta_1}{\partial u}\right)\frac{\tau_1}{\sigma_1}\right]\frac{\partial \theta_1'}{\partial v} = 0. \end{aligned}$$

The adjoint of this equation is obtained by replacing ϕ_1' in (16) by ϕ_1'/w_1.

We consider the transformations for which ϕ_1 is any solution of (2) and $\theta_1 = 1$. The corresponding functions σ_1 and τ_1 are given by

$$\tag{23} \begin{aligned} \sigma_1' &= \int b\phi_1\, du + \left(\frac{\partial \phi_1}{\partial v} - a\phi_1\right) dv, \\ \tau_1' &= \int \left(\frac{\partial \phi_1}{\partial u} - b\phi_1\right) du + a\phi_1\, dv. \end{aligned}$$

It is readily shown that these particular values are in the following relations with the functions σ_1 and τ_1 as given by (4) and (6):

$$\tag{24} \tau_1' = \frac{\tau_1}{\theta_1} - w_1, \qquad \sigma_1' = \frac{\sigma_1}{\theta_1} + w_1.$$

If $S^{(1)}$ denotes the corresponding transform of S, and its cartesian coördinates are denoted by $x^{(1)}, y^{(1)}, z^{(1)}$, we have from (19) and (20)

$$\tag{25} \frac{\partial x^{(1)}}{\partial u} = \tau_1' \frac{\partial x}{\partial u}, \qquad \frac{\partial x^{(1)}}{\partial v} = -\sigma_1' \frac{\partial x}{\partial v}.$$

From the form of these equations it is evident that the tangent planes to S and $S^{(1)}$ at corresponding points are parallel. Hence, when $\theta_1 = 1$, we have the *parallel* transformations T.

4. Theorem of permutability of transformations T

Suppose that we have two transforms S_1 and S_2 of S determined by the respective sets of functions σ_1, τ_1 and σ_2, τ_2, where σ_2 and τ_2 are given by (4) and (6) when θ_1 and ϕ_1 are replaced by θ_2 and ϕ_2. A solution θ_{12} of equation (22) is given by the quadratures

$$\tag{26} \frac{\partial}{\partial u}(w_1\,\theta_{12}) = \tau_1 \frac{\partial}{\partial u}\left(\frac{\theta_2}{\theta_1}\right), \qquad \frac{\partial}{\partial v}(w_1\,\theta_{12}) = -\sigma_1 \frac{\partial}{\partial v}\left(\frac{\theta_2}{\theta_1}\right),$$

which are of the same form as (20).

We consider the surface S_{12} obtained from S_1 by the transformation T determined by θ_{12} and ϕ_{12}, as given by (17), where now

$$\phi_1^{-1} = \frac{w_1\,\theta_1\,\phi_1}{\tau_1\,\sigma_1}. \tag{27}$$

The function w_{12} of this transformation is given by

$$\frac{\partial w_{12}}{\partial u} = \tau_{12}\frac{\partial}{\partial u}\left(\frac{1}{\theta_{12}}\right), \qquad \frac{\partial w_{12}}{\partial v} = -\,\sigma_{12}\frac{\partial}{\partial v}\left(\frac{1}{\theta_{12}}\right), \tag{28}$$

where σ_{12} and τ_{12} are defined by equations analogous to (4) and (6), namely

$$\begin{aligned}\sigma_{12} &= \int \phi_{12}\left[\frac{\partial\theta_{12}}{\partial u} + \theta_{12}\left\{\frac{\partial \log w_1}{\partial u} - \left(b + \frac{\partial \log \theta_1}{\partial u}\right)\frac{\tau_1}{\sigma_1}\right\}\right]du \\ &\qquad + \theta_{12}\left[\frac{\partial\phi_{12}}{\partial v} - \phi_{12}\left\{\frac{\partial \log w_1}{\partial v} - \left(a + \frac{\partial \log \theta_1}{\partial v}\right)\frac{\sigma_1}{\tau_1}\right\}\right]dv, \\ \tau_{12} &= \int \theta_{12}\left[\frac{\partial\phi_{12}}{\partial u} - \phi_{12}\left\{\frac{\partial \log w_1}{\partial u} - \left(b + \frac{\partial \log \theta_1}{\partial u}\right)\frac{\tau_1}{\sigma_1}\right\}\right]du \\ &\qquad + \phi_{12}\left[\frac{\partial\theta_{12}}{\partial v} + \theta_{12}\left\{\frac{\partial \log w_1}{\partial v} - \left(a + \frac{\partial \log \theta_1}{\partial v}\right)\frac{\sigma_1}{\tau_1}\right\}\right]dv.\end{aligned} \tag{29}$$

By means of (17), (26), and (27) these expressions are reducible to

$$\begin{aligned}\sigma_{12} &= \int \phi_{12}\left[-\frac{\tau_1}{w_1}\frac{\partial}{\partial u}\left(\frac{\theta_2}{\theta_1}\right) - \left(b + \frac{\partial \log \theta_1}{\partial u}\right)\frac{\theta_{12}\,\tau_1}{\sigma_1}\right]du \\ &\qquad + \theta_{12}\left[\frac{\phi_{12}\,\tau_1}{\sigma_1\,\phi_1}\left(a\phi_1 - \frac{\partial\phi_1}{\partial v}\right) + \frac{\theta_1\,\phi_1\,w_1}{\sigma_1}\frac{\partial}{\partial v}\left(\frac{\phi_2}{\phi_1}\right)\right]dv, \\ \tau_{12} &= \int \theta_{12}\left[\frac{\phi_{12}\,\sigma_1}{\tau_1\,\phi_1}\left(b\phi_1 - \frac{\partial\phi_1}{\partial u}\right) - \frac{\theta_1\,\phi_1\,w_1}{\tau_1}\frac{\partial}{\partial u}\left(\frac{\phi_2}{\phi_1}\right)\right]du \\ &\qquad + \phi_{12}\left[\frac{\sigma_1}{w_1}\frac{\partial}{\partial v}\left(\frac{\theta_2}{\theta_1}\right) - \left(a + \frac{\partial \log \theta_1}{\partial v}\right)\frac{\theta_{12}\,\sigma_1}{\tau_1}\right]dv.\end{aligned} \tag{30}$$

The coördinates x_{12}, y_{12}, z_{12} of S_{12} are given by equations similar to (21), which are reducible by (26) to

$$\begin{aligned}\frac{\partial x_{12}}{\partial u} &= \frac{\tau_1\,\tau_{12}}{\theta_1^2\,\theta_{12}^2\,w_1\,w_{12}}\left[x_{12}\left\{\theta_{12}\frac{\partial\theta_1}{\partial u} + \theta_1^2\frac{\partial}{\partial u}\left(\frac{\theta_2}{\theta_1}\right)\right\}\right. \\ &\qquad \left. - x_1\,\theta_1^2\frac{\partial}{\partial u}\left(\frac{\theta_2}{\theta_1}\right) + \theta_{12}\,\theta_1^2\frac{\partial}{\partial u}\left(\frac{x}{\theta_1}\right)\right], \\ \frac{\partial x_{12}}{\partial v} &= \frac{\sigma_1\,\sigma_{12}}{\theta_1^2\,\theta_{12}^2\,w_1\,w_{12}}\left[x_{12}\left\{\theta_{12}\frac{\partial\theta_1}{\partial v} + \theta_1^2\frac{\partial}{\partial v}\left(\frac{\theta_2}{\theta_1}\right)\right\}\right. \\ &\qquad \left. - x_1\,\theta_1^2\frac{\partial}{\partial v}\left(\frac{\theta_2}{\theta_1}\right) + \theta_{12}\,\theta_1^2\frac{\partial}{\partial v}\left(\frac{x}{\theta_1}\right)\right].\end{aligned} \tag{31}$$

Proceeding in like manner with S_2, we obtain a transform S_{21} by means of functions θ_{21}, ϕ_{21}, σ_{21}, τ_{21}. The equations similar to (31) are obtained by interchanging the subscripts 1 and 2 in (31). From these two sets of equations it follows that S_{12} and S_{21} are the same surface, if

$$\begin{aligned} \tau_{12}\,\tau_1\,\theta_2\,\theta_{21}\,w_2\,w_{21} &= \tau_{21}\,\tau_2\,\theta_1\,\theta_{12}\,w_1\,w_{12}, \\ \sigma_{12}\,\sigma_1\,\theta_2\,\theta_{21}\,w_2\,w_{21} &= \sigma_{21}\,\sigma_2\,\theta_1\,\theta_{12}\,w_1\,w_{12}, \end{aligned} \tag{32}$$

and

$$x_{12} = \Theta_{12}\,(x_1\,\theta_2\,\theta_{21} + x_2\,\theta_1\,\theta_{12} - x\,\theta_{12}\,\theta_{21}), \tag{33}$$

where

$$\frac{1}{\Theta_{12}} = \theta_2\,\theta_{21} + \theta_1\,\theta_{12} - \theta_{12}\,\theta_{21}. \tag{34}$$

When we express the condition that this value of x_{12} shall satisfy equations (31), we get

$$\begin{aligned} \left[\tau_{12}\,\tau_1 - \theta_1\,\theta_{12}\,w_1\,w_{12}\,\Theta_{12}\left(\frac{\theta_2\,\theta_{21}\,\tau_1}{\theta_1\,w_1} + \frac{\theta_1\,\theta_{12}\,\tau_2}{\theta_2\,w_2} - \theta_{12}\,\theta_{21}\right)\right]\Bigg[x_1\Bigg(-\theta_{21}\frac{\partial\theta_2}{\partial u} \\ - \theta_2\frac{\partial\theta_1}{\partial u} + \theta_1\frac{\partial\theta_2}{\partial u}\Bigg) + x_2\left(\theta_{12}\frac{\partial\theta_1}{\partial u} + \theta_2\frac{\partial\theta_1}{\partial u} - \theta_1\frac{\partial\theta_2}{\partial u}\right) \\ - x\left(\theta_{21}\frac{\partial\theta_2}{\partial u} + \theta_{12}\frac{\partial\theta_1}{\partial u}\right) + \frac{1}{\Theta_{12}}\frac{\partial x}{\partial u}\Bigg] = 0, \end{aligned}$$

$$\begin{aligned} \left[\sigma_{12}\,\sigma_1 + \theta_1\,\theta_{12}\,w_1\,w_{12}\,\Theta_{12}\left(\frac{\theta_2\,\theta_{21}\,\sigma_1}{\theta_1\,w_1} + \frac{\theta_1\,\theta_{12}\,\sigma_2}{\theta_2\,w_2} + \theta_{12}\,\theta_{21}\right)\right]\Bigg[x_1\Bigg(-\theta_{21}\frac{\partial\theta_2}{\partial v} \\ - \theta_2\frac{\partial\theta_1}{\partial v} + \theta_1\frac{\partial\theta_2}{\partial v}\Bigg) + x_2\left(\theta_{12}\frac{\partial\theta_1}{\partial v} + \theta_2\frac{\partial\theta_1}{\partial v} - \theta_1\frac{\partial\theta_2}{\partial v}\right) \\ - x\left(\theta_{21}\frac{\partial\theta_2}{\partial v} + \theta_{12}\frac{\partial\theta_1}{\partial v}\right) + \frac{1}{\Theta_{12}}\frac{\partial x}{\partial v}\Bigg] = 0. \end{aligned}$$

On the assumption that θ_1 and θ_2 are independent, these are equivalent to

$$\begin{aligned} \sigma_1\,\sigma_{12} &= -\theta_1\,\theta_{12}\,w_1\,w_{12}\,\Theta_{12}\left(\frac{\theta_2\,\theta_{21}\,\sigma_1}{\theta_1\,w_1} + \frac{\theta_1\,\theta_{12}\,\sigma_2}{\theta_2\,w_2} + \theta_{12}\,\theta_{21}\right), \\ \tau_1\,\tau_{12} &= \theta_1\,\theta_{12}\,w_1\,w_{12}\,\Theta_{12}\left(\frac{\theta_2\,\theta_{21}\,\tau_1}{\theta_1\,w_1} + \frac{\theta_1\,\theta_{12}\,\tau_2}{\theta_2\,w_2} - \theta_{12}\,\theta_{21}\right). \end{aligned}$$

From these equations and one analogous to (8) we obtain

$$\phi_{12} = \frac{\theta_1^2\,\theta_{12}\,w_1\,w_{12}}{\sigma_1\,\tau_1}\Theta_{12}\left(-\phi_1\,\theta_{21} + (\sigma_1\,\tau_2 - \sigma_2\,\tau_1)\frac{1}{\theta_2\,w_2}\right).$$

When this value is substituted in (17), we find that the transformation func-

tions must have the expressions

$$
(35)\qquad
\begin{aligned}
w_{12} &= \frac{w_2}{\theta_1\,\theta_{12}\,\Theta_{12}},\\
\phi_{12} &= \frac{w_1\,\theta_1\,\phi_1}{\tau_1\,\sigma_1}\left(-\,w_2\,\theta_{21} + \frac{1}{\theta_2\,\phi_1}(\tau_2\,\sigma_1 - \tau_1\,\sigma_2)\right),\\
\sigma_{12} &= -\,\frac{w_1\,w_2}{\sigma_1}\left(\frac{\theta_2\,\theta_{21}\,\sigma_1}{\theta_1\,w_1} + \frac{\theta_1\,\theta_{12}\,\sigma_2}{\theta_2\,w_2} + \theta_{12}\,\theta_{21}\right),\\
\tau_{12} &= \frac{w_1\,w_2}{\tau_1}\left(\frac{\theta_2\,\theta_{21}\,\tau_1}{\theta_1\,w_1} + \frac{\theta_1\,\theta_{12}\,\tau_2}{\theta_2\,w_2} - \theta_{12}\,\theta_{21}\right).
\end{aligned}
$$

It is readily verified that these values for σ_{12} and τ_{12} satisfy equations (30).

In consequence of (35) equation (33) can be written

$$
(36)\qquad \theta_1\,\theta_{12}\,w_{12}\,x_{12} = w_2\,(\theta_2\,\theta_{21}\,x_1 + \theta_1\,\theta_{12}\,x_2 - \theta_{12}\,\theta_{21}\,x)\,.
$$

Thus we have established a theorem of permutability of general transformations T. There are two arbitrary constants involved, namely in the determination of θ_{12} by (26) and of θ_{21} by

$$
(37)\qquad \frac{\partial}{\partial u}(w_2\,\theta_{21}) = \tau_2\frac{\partial}{\partial u}\left(\frac{\theta_1}{\theta_2}\right),\qquad \frac{\partial}{\partial v}(w_2\,\theta_{21}) = -\,\sigma_2\frac{\partial}{\partial v}\left(\frac{\theta_1}{\theta_2}\right).
$$

Accordingly we formulate

THEOREM 1. *If S_1 and S_2 are two transforms of S, there exist ∞^2 surfaces S_{12}, each of which is a transform of both S_1 and S_2; and their complete determination requires two quadratures.*

We say that four such surfaces S, S_1, S_2, S_{12} form a *quatern.*

We consider, in particular, the case where S_2 is parallel to S. If we take $\theta_2 = 1$, in accordance with (19) and (26) we have $\theta_{12} = 1$ as one solution. Now (33) becomes

$$
(38)\qquad (x_{12} - x_2)\,\theta_1 = (x_1 - x)\,\theta_{21}\,.
$$

Hence we have

THEOREM 2. *If S_2 is parallel to S and S_1 is any transform of S, one of the surfaces S_{12} is parallel to S_1; moreover the lines joining corresponding points on S_{12} and S_2 and on S and S_1 are parallel.*

If both S_1 and S_2 are parallel to S, the functions θ_1 and θ_2 are constants. Hence from (33) it follows that x_{12} is a linear function of x, x_1, x_2, with constant coefficients, and consequently S_{12} also is parallel to S.

5. Envelope of the planes of a quatern

If M, M_1, M_2, M_{12} are corresponding points of four surfaces of a quatern, it follows from (35) and (36) that these four points lie in a plane π. Since

this plane contains the lines MM_1 and MM_2 which generate congruences conjugate to the parametric conjugate system on S, it envelopes a surface Σ upon which the parametric curves form a conjugate system, as follows from the general theory of congruences.* Moreover, if Π is the point of the envelope corresponding to M on S, the tangent at Π to one of these curves passes through the focal points F_1' and F_2' of the lines MM_1 and MM_2 respectively, and the tangent to the other curve passes through the focal points F_1'' and F_2''. We will now find the coördinates of Π.

In cartesian coördinates equations (5) and (7) are of the form

$$(39) \qquad w_1 x_1 = x_1' \sigma_1' - \frac{\sigma_1}{\theta_1} x, \qquad w_1 x_1 = - x_1'' \tau_1' + \frac{\tau_1}{\theta_1} x,$$

where now $x_1' \sigma_1'$ and $x_1'' \tau_1'$ are respectively equal to the right-hand members of (3). Similar equations with subscripts 2 hold for the congruence of lines MM_2.

The cartesian coördinates ξ, η, ζ of Π are given by equations of the form

$$\begin{aligned} \xi &= \frac{1}{\sigma_1'}\left(x_1 w_1 + \frac{\sigma_1}{\theta_1} x\right) + t_1\left[\frac{1}{\sigma_1'}\left(x_1 w_1 + \frac{\sigma_1}{\theta_1} x\right) - \frac{1}{\sigma_2}\left(x_2 w_2 + \frac{\sigma_2}{\theta_2} x\right)\right], \\ (40) \qquad &= \frac{1}{\tau_1'}\left(- x_1 w_1 + \frac{\tau_1}{\theta_1} x\right) + t_2\left[\frac{1}{\tau_1'}\left(- x_1 w_1 + \frac{\tau_1}{\theta_1} x\right)\right. \\ &\qquad \left. - \frac{1}{\tau_2'}\left(- x_2 w_2 + \frac{\tau_2}{\theta_2} x\right)\right], \end{aligned}$$

where t_1 and t_2 are to be determined. When these two expressions for ξ are equated, we get an equation of the form

$$Ax + Bx_1 + Cx_2 = 0,$$

where A, B, and C are determinate functions. Since similar equations in the y's and z's also must hold, we must have $A = B = C = 0$. From the first two of these equations we get

$$\phi_1\left(w_2 + \frac{\sigma_2}{\theta_2}\right) + t_1\left(w_2 \phi_1 - w_1 \phi_2 + \frac{\sigma_2 \tau_1 - \sigma_1 \tau_2}{\theta_1 \theta_2}\right) = 0,$$

$$\phi_1\left(w_2 - \frac{\tau_2}{\theta_2}\right) + t_2\left(w_2 \phi_1 - w_1 \phi_2 + \frac{\sigma_2 \tau_1 - \sigma_1 \tau_2}{\theta_1 \theta_2}\right) = 0.$$

These values satisfy $C = 0$, and when substituted in the above expressions

---

*Guichard, Annales de l'école normale superieure, ser. 3, vol. 14 (1897).

for ξ we get

$$\xi\left(w_2\,\phi_1 - w_1\,\phi_2 + \frac{\sigma_2\,\tau_1 - \sigma_1\,\tau_2}{\theta_1\,\theta_2}\right) = w_2\,\phi_1\,x_2 - w_1\,\phi_2\,x_1 - \frac{\sigma_1\,\tau_2 - \sigma_2\,\tau_1}{\theta_1\,\theta_2}\,x\,. \tag{41}$$

We shall find the functions of the theorem of permutability when homogeneous coördinates are used. Now the functions θ_{12} and θ_{21} are given by

$$\frac{\partial\theta_{ij}}{\partial u} = \tau_i\frac{\partial}{\partial u}\left(\frac{\theta_j}{\theta_i}\right), \qquad \frac{\partial\theta_{ij}}{\partial v} = -\,\sigma_i\frac{\partial}{\partial v}\left(\frac{\theta_j}{\theta_i}\right) \qquad \left(\begin{matrix} i = 1, 2, \\ j = 1, 2, \end{matrix}\; i \neq j\right), \tag{42}$$

and the coördinates $x_{12}, \cdots, w_{12}$ of S_{12} must satisfy the equations of the form

$$\frac{\partial x_{ij}}{\partial u} = \tau_{ij}\frac{\partial}{\partial u}\left(\frac{x_i}{\theta_{ij}}\right), \qquad \frac{\partial x_{ij}}{\partial v} = -\,\sigma_{ij}\frac{\partial}{\partial v}\left(\frac{x_i}{\theta_{ij}}\right) \qquad \left(\begin{matrix} i = 1, 2, \\ j = 1, 2, \end{matrix}\; i \neq j\right). \tag{43}$$

From (35) it follows that the functions $\tau_{12}, \sigma_{12}, \phi_{12}$ are of the form

$$\begin{aligned}
\tau_1\,\tau_{12} &= \tau_2\,\tau_{21} = \frac{\theta_2\,\theta_{21}\,\tau_1}{\theta_1} + \frac{\theta_1\,\theta_{12}\,\tau_2}{\theta_2} - \theta_{12}\,\theta_{21}, \\
\sigma_1\,\sigma_{12} &= \sigma_2\,\sigma_{21} = -\left(\frac{\theta_2\,\theta_{21}\,\sigma_1}{\theta_1} + \frac{\theta_1\,\theta_{12}\,\sigma_2}{\theta_2} + \theta_{12}\,\theta_{21}\right), \\
\phi_{12} &= \frac{\theta_1\,\phi_1}{\sigma_1\,\tau_1}\left(-\,\theta_{21} + (\sigma_1\,\tau_2 - \sigma_2\,\tau_1)\frac{1}{\theta_2\,\phi_1}\right).
\end{aligned} \tag{44}$$

Moreover, the coördinate x_{12} is expressed by

$$\theta_1\,\theta_{12}\,x_{12} = \theta_2\,\theta_{21}\,x_1 + \theta_1\,\theta_{12}\,x_2 - \theta_{12}\,\theta_{21}\,x\,. \tag{45}$$

6. Transformations T determined by the same function ϕ

We consider now the relation of two transformations determined by θ_1 and θ_2 respectively but by the same function ϕ_1. If we put

$$\begin{aligned}
(\tau_1)_2 &= \int \theta_2\left(\frac{\partial\phi_1}{\partial u} - b\phi_1\right)du + \phi_1\left(\frac{\partial\theta_2}{\partial v} + a\theta_2\right)dv, \\
(\sigma_1)_2 &= \int \phi_1\left(\frac{\partial\theta_2}{\partial u} + b\theta_2\right)du + \theta_2\left(\frac{\partial\phi_1}{\partial v} - a\phi_1\right)dv,
\end{aligned}$$

we have in consequence of (26)

$$(\tau_1)_2 = \frac{\theta_2}{\theta_1}\tau_1 - w_1\,\theta_{12}, \qquad (\sigma_1)_2 = \frac{\theta_2}{\theta_1}\sigma_1 + w_1\,\theta_{12}\,. \tag{46}$$

When these values are substituted in equations analogous to (20), namely

$$\frac{\partial}{\partial u}(x_2\,w_2) = (\tau_1)_2\frac{\partial}{\partial u}\left(\frac{x}{\theta_2}\right), \qquad \frac{\partial}{\partial v}(x_2\,w_2) = -\,(\sigma_1)_2\frac{\partial}{\partial v}\left(\frac{x}{\theta_2}\right),$$

the latter can be integrated in the form

$$(47) \qquad x_2 w_2 = x_1 w_1 - x w_1 \frac{\theta_{12}}{\theta_2} .$$

In like manner from equations analogous to (19) we get

$$(48) \qquad w_2 = \frac{w_1}{\theta_2} (\theta_2 - \theta_{12}) + c ,$$

where c is a constant. By means of this result equation (47) can be given the form

$$(49) \qquad (x_2 - x) w_2 = (x_1 - x) w_1 - cx .$$

From this it is seen that the congruence G_2 of lines joining corresponding points on S and S_2 is the same as the congruence G_1 only in case $c = 0$ in (48).

When the above expressions for $(\sigma_1)_2$ and $(\tau_1)_2$ are substituted in (37), we find

$$(50) \qquad w_2 \theta_{21} = - w_1 \theta_{12} \frac{\theta_1}{\theta_2} + k ,$$

where k is an additive constant.

From (33) and (34) we obtain for the present case

$$(51) \qquad \begin{gathered} \frac{w_2}{\Theta_{12}} = (\theta_2 - \theta_{12}) k + \theta_1 \theta_{12} c , \qquad w_{12} = \frac{(\theta_2 - \theta_{12}) k}{\theta_1 \theta_{12}} + c , \\ x_{12} (k (\theta_2 - \theta_{12}) + c \theta_1 \theta_{12}) = k \frac{w_2 \theta_2}{w_1} x_2 . \end{gathered}$$

Hence if $k = 0$, the surface S_{12} reduces to a point; if $c = 0$, it coincides with S_2.

In the inverse transformation from S_1 to S the function w_1^{-1} has the value $1/\theta_1$, as is evident from (20). If we look upon S and S_{12} as transforms of S_1, the analogue of equation (47) is

$$(52) \qquad x_{12} w_{12} = w_1^{-1} \left(x - x_1 \frac{\theta_2}{\theta_{12}} \right) .$$

This equation is satisfied by the value of x_{12}, given by (51), provided $k = -1$.

Incidentally we remark that the last of (51) can be written

$$(53) \qquad x_{12} = x_2 \Big/ \left(1 - \frac{c}{k} \theta_{21} \right) .$$

The denominator of this equation is a solution of the point equation of S_2. Moreover, corresponding points on S_2 and S_{12} are on a line through the origin. This is a type of transformations which we will consider later (§ 13); we call them *radial* transformations.

Accordingly we have

THEOREM 3. *When a transform S_1 of S is known, the determination of another transform S_2 with the same function ϕ requires a single quadrature; then the fourth surface of the quatern is a radial transform of S_2.*

7. ANOTHER FORM OF TRANSFORMATIONS T

Particular importance attaches to the results of the preceding section when we take a parallel surface for S_2. As in § 3, we call it $S^{(1)}$ and its coördinates $x^{(1)}, y^{(1)}, z^{(1)}$. We take $\theta_2 = 1$, then $\theta_{12} = 1$. Also we take $c = 1$. Then (47) assumes the desired form

$$x_1 = x + \frac{x^{(1)}}{w_1}. \tag{54}$$

In consequence of (24) equations (19) can be written

$$\frac{\partial}{\partial u}(w_1 \theta_1) = -\tau_1' \frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial}{\partial v}(w_1 \theta_1) = \sigma_1' \frac{\partial \theta_1}{\partial v}.$$

Hence if we put

$$w_1 \theta_1 = -\theta_1^{(1)}, \tag{55}$$

we have

$$\frac{\partial \theta_1^{(1)}}{\partial u} = \tau_1' \frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial \theta_1^{(1)}}{\partial v} = -\sigma_1' \frac{\partial \theta_1}{\partial v}, \tag{56}$$

which are similar to (25). By means of (55) equation (54) is reducible to*

$$x_1 = x - \frac{\theta_1}{\theta_1^{(1)}} x^{(1)}. \tag{57}$$

The significance of this result is that the problem of finding transformations T is reduced to that of finding parallel transforms and the integration of equation (1).

Equations (57) enable us to show that when we take

$$\theta_1 = ax + by + cz, \qquad \theta_1^{(1)} = ax^{(1)} + by^{(1)} + cz^{(1)},$$

where a, b, c are constants, then S_1 is the plane $ax_1 + by_1 + cz_1 = 0$.

In consequence of (24) and (54) equations (39) giving the coördinates of the focal points of the congruence G_1 are reducible to

$$x_1' = x + \frac{x^{(1)}}{\sigma_1'}, \qquad x_1'' = x - \frac{x^{(1)}}{\tau_1'}. \tag{58}$$

As an application of these results we seek the condition that S_1 shall be normal to the lines of the congruence G_1. From (57) it is seen that $x^{(1)}$, $y^{(1)}$, $z^{(1)}$ are the direction-parameters of the lines of this congruence. Hence

* Cf. Jonas, l. c., p. 102.

we must have

$$\sum x^{(1)}\frac{\partial x_1}{\partial u}=0,\qquad \sum x^{(1)}\frac{\partial x_1}{\partial v}=0.$$

Substituting the values of x_1, y_1, z_1 from (57), we have to within a constant factor

$$\theta_1^{(1)}=\sqrt{x^{(1)^2}+y^{(1)^2}+z^{(1)^2}}. \tag{59}$$

From (22) it follows that the point equation of $S^{(1)}$ is

$$\frac{\partial^2\theta_1^{(1)}}{\partial u\partial v}-a\frac{\sigma_1'}{\tau_1'}\frac{\partial\theta_1^{(1)}}{\partial u}-b\frac{\tau_1'}{\sigma_1'}\frac{\partial\theta_1^{(1)}}{\partial v}=0.$$

Expressing the condition that the above value of $\theta_1^{(1)}$ satisfies this equation, we get

$$\sum\frac{\partial x^{(1)}}{\partial u}\frac{\partial x^{(1)}}{\partial v}-\frac{\partial\theta_1^{(1)}}{\partial u}\frac{\partial\theta_1^{(1)}}{\partial v}=0,$$

and consequently

$$\sum\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}-\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_1}{\partial v}=0.$$

But this is the condition that $x^2+y^2+z^2-\theta_1^2$ also is a solution of the point equation of S. Moreover, it follows from (59) and (57) that in this case θ_1 is the distance from S to S_1. Hence from this point of view we have established the known

THEOREM 4. *When the developables of the congruence of normals to a surface S_1 meet a surface S in a conjugate system, the function t giving the distance between corresponding points on S and S_1 is a solution of the point equation of S as is also the function $x^2+y^2+z^2-t^2$.*

Returning to the general case, we have from (50) in consequence of (55)

$$\theta_{21}=\theta_1^{(1)}-1. \tag{60}$$

We have taken $k=-1$ so that (52) shall hold. Then S_{12} is the parallel $S_1^{(1)}$ of S_1 by means of which S is obtained from S_1. Consequently the present form of (53) is

$$x_1^{(1)}=\frac{x^{(1)}}{\theta_1^{(1)}}. \tag{61}$$

8. Another form of the equations of the theorem of permutability

Suppose now that we have two transforms S_1 and S_2 of S; we wish to give the theorem of permutability a new form in view of the preceding results. Evidently the functions θ_{12} and θ_{21} are given by expressions analogous to (57), namely

$$\theta_{12}=\theta_2-\frac{\theta_1}{\theta_1^{(1)}}\theta_2^{(1)}\qquad \theta_{21}=\theta_1-\frac{\theta_2}{\theta_2^{(2)}}\theta_1^{(2)}, \tag{62}$$

where $\theta_i^{(j)}$ is defined by

$$\frac{\partial \theta_i^{(j)}}{\partial u} = \tau_j' \frac{\partial \theta_i}{\partial u}, \qquad \frac{\partial \theta_i^{(j)}}{\partial v} = -\sigma_j' \frac{\partial \theta_i}{\partial v}, \tag{63}$$

σ_j' and τ_j' being given by equations obtained from (23) on replacing 1 by j. Now

$$\frac{1}{\Theta_{12}} = \frac{\theta_1 \theta_2}{\theta_1^{(1)} \theta_2^{(2)}} (\theta_1^{(1)} \theta_2^{(2)} - \theta_2^{(1)} \theta_1^{(2)}),$$

and from (33) we have

$$\begin{aligned}(\theta_1^{(1)} \theta_2^{(2)} - \theta_2^{(1)} \theta_1^{(2)}) (x_{12} - x) \\ = (\theta_1^{(2)} \theta_2 - \theta_2^{(2)} \theta_1) x^{(1)} + (\theta_2^{(1)} \theta_1 - \theta_1^{(1)} \theta_2) x^{(2)}.\end{aligned} \tag{64}$$

From this equation and (57) we obtain

$$(\theta_1^{(1)} \theta_2^{(2)} - \theta_2^{(1)} \theta_1^{(2)}) (x_{12} - x_1) = (\theta_1^{(1)} \theta_2 - \theta_2^{(1)} \theta_1) \left(\frac{\theta_1^{(2)}}{\theta_1^{(1)}} x^{(1)} - x^{(2)} \right). \tag{65}$$

We note that the expression in the last parenthesis is similar in form to the right-hand member of (57). Hence if we put

$$x_1^{(2)} = x^{(2)} - \frac{\theta_1^{(2)}}{\theta_1^{(1)}} x^{(1)}, \tag{66}$$

the surface $S_1^{(2)}$ whose coördinates are $x_1^{(2)}$, $y_1^{(2)}$, $z_1^{(2)}$ is a transform of $S^{(2)}$.

If equation (66) be differentiated, the resulting equations are reducible to

$$\begin{aligned}\frac{\partial x_1^{(2)}}{\partial u} &= \left(\tau_2' - \frac{\theta_1^{(2)}}{\theta_1^{(1)}} \tau_1' \right) \frac{\theta_1 w_1}{\tau_1} \frac{\partial x_1}{\partial u}, \\ \frac{\partial x_1^{(2)}}{\partial v} &= \left(\sigma_2' - \frac{\theta_1^{(2)}}{\theta_1^{(1)}} \sigma_1' \right) \frac{\theta_1 w_1}{\sigma_1} \frac{\partial x_1}{\partial v}.\end{aligned} \tag{67}$$

Hence $S_1^{(2)}$ is parallel to S_1. We wish to show that it is the parallel surface whose coördinates enable the equations of the transformation from S_1 to S_{12} to be given a form similar to (57). The first derivatives of the coördinates of this desired parallel surface are equal to

$$\tau_{12}' \frac{\partial x_1}{\partial u}, \qquad -\sigma_{12}' \frac{\partial x_1}{\partial v},$$

where in consequence of (24), (35), (57), and (55)

$$\begin{aligned}\tau_{12}' &= \frac{\tau_{12}}{\theta_{12}} - w_{12} = \frac{w_1 \theta_1}{\tau_1} \left(\tau_2' - \frac{\theta_1^{(2)}}{\theta_1^{(1)}} \tau_1' \right), \\ \sigma_{12}' &= \frac{\sigma_{12}}{\theta_{12}} + w_{12} = -\frac{w_1 \theta_1}{\sigma_1} \left(\sigma_2' - \frac{\theta_1^{(2)}}{\theta_1^{(1)}} \sigma_1' \right).\end{aligned} \tag{68}$$

Comparing these results with (67) we find that $S_1^{(2)}$ is the desired surface.

If we write (65) in the form

$$x_{12} = x_1 - \frac{\theta_2 - \frac{\theta_1}{\theta_1^{(1)}}\theta_2^{(1)}}{\theta_2^{(2)} - \frac{\theta_1^{(2)}}{\theta_1^{(1)}}\theta_2^{(1)}} x_1^{(2)}, \tag{69}$$

we note its similarity to (57).

9. Transformations K

We consider now the particular conjugate systems for which the invariants h and k of the point equation are equal. From (14) it is seen that in this case the point equation may be written

$$\frac{\partial^2 \theta}{\partial u \partial v} + \frac{\partial \log \sqrt{\rho}}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u}\frac{\partial \theta}{\partial v} = 0. \tag{70}$$

From (22) it is seen that the parametric system on S_1 will have equal invariants, if σ_1 and τ_1 are equal. From (4) and (6) it follows that to within a constant factor we must have

$$\phi_1 = 2\theta_1 \rho. \tag{71}$$

It is readily verified that this value of ϕ_1 satisfies the equation adjoint to (70). From (8) we have

$$\sigma_1 = \tau_1 = \theta_1^2 \rho. \tag{72}$$

Now equations (21) become

$$\begin{aligned} \frac{\partial x_1}{\partial u} &= \frac{\rho}{w_1}\left(\theta_1 \frac{\partial x}{\partial u} + (x_1 - x)\frac{\partial \theta_1}{\partial u}\right), \\ \frac{\partial x_1}{\partial v} &= -\frac{\rho}{w_1}\left(\theta_1 \frac{\partial x}{\partial v} + (x_1 - x)\frac{\partial \theta_1}{\partial v}\right), \end{aligned} \tag{73}$$

where w_1 is given by

$$\frac{\partial w_1}{\partial u} = -\rho\frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial w_1}{\partial v} = \rho\frac{\partial \theta_1}{\partial v}. \tag{74}$$

This transformation is the same which we have considered at length in a former paper, and called a transformation K.*

Equations (24) reduce to

$$\sigma_1' = \rho\theta_1 + w_1, \qquad \tau_1' = \rho\theta_1 - w_1,$$

and consequently the expressions (39) for the coördinates of the focal points

* M_1, p. 400.

of the congruence G_1 become

$$x_1' = \frac{\rho\theta_1 x + w_1 x_1}{\rho\theta_1 + w_1}, \qquad x_1'' = \frac{\rho\theta_1 x - w_1 x_1}{\rho\theta_1 - w_1}.$$

The foregoing results are stated in

THEOREM 5. *When a surface S is referred to a conjugate system with equal point invariants and θ_1 is any solution of the point equation of S, the surface S_1 whose coördinates are given by quadratures of the form* (73), *is referred to a conjugate system with equal point invariants, and the developables of the lines jo ning corresponding points on S and S_1 meet these surfaces in these parametric curves. Moreover, the focal points of the congruence are harmonic to the corresponding points on S and S_1.*

We assume that S_1 and S_2 are two surfaces in the relation of transformations K with S, and apply the theorem of permutability. Equations (26) and (37) are now reducible to

$$(75) \qquad \begin{aligned} \frac{\partial}{\partial u}(w_i\,\theta_{ij}) &= \rho\left(\theta_i\frac{\partial\theta_j}{\partial u} - \theta_j\frac{\partial\theta_i}{\partial u}\right), \\ \frac{\partial}{\partial v}(w_i\,\theta_{ij}) &= -\rho\left(\theta_i\frac{\partial\theta_j}{\partial v} - \theta_j\frac{\partial\theta_i}{\partial v}\right) \end{aligned} \qquad \left(\begin{matrix} i = 1, 2, \\ j = 1, 2, \end{matrix}\quad i \neq j\right),$$

from which it follows that

$$w_2\,\theta_{21} + w_1\,\theta_{12} = \text{const.}$$

From (35) it is seen that the necessary and sufficient condition that $\sigma_{12} = \tau_{12}$ is

$$(76) \qquad w_2\,\theta_{21} + w_1\,\theta_{12} = 0.$$

Hence we have

THEOREM 6. *When S_1 and S_2 are two surfaces in the relation of transformations K with a surface S, of the ∞^2 surfaces S_{12} forming quaterns with them in accordance with the theorem of permutability of transformations T, ∞^1 are in the relation of transformations K with S_1 and S_2.*

From (35) we have

$$(77) \quad \theta_1\,w_{12} = w_1(\theta_{12} - \theta_2) + w_2\,\theta_1, \qquad \theta_2\,w_{21} = w_2(\theta_{21} - \theta_1) + w_1\,\theta_2,$$

and from (36)

$$(78) \qquad \theta_1\,w_{12}\,x_{12} = -w_1\,\theta_2\,x_1 + w_2\,\theta_1\,x_2 + w_1\,\theta_{12}\,x.$$

The coördinates ξ, η, ζ, of Π, the point of contact of the plane π with its envelope, as given by (41), are expressible in the form

$$(79) \qquad \xi = -\frac{w_2\,\theta_1\,x_2 - w_1\,\theta_2\,x_1}{w_2\,\theta_1 - w_1\,\theta_2} = -\frac{\theta_1\,w_{12}\,x_{12} - w_1\,\theta_{12}\,x}{\theta_1\,w_{12} - w_1\,\theta_{12}}.$$

Hence Π is the intersection of the lines MM_{12} and $M_1 M_2$; consequently the points M_{12} of the ∞^1 surfaces S_{12} lie on the line MM_{12}. Therefore we have

THEOREM 7. *If S, S_1, S_2, S_{12} are four surfaces of a quatern for transformations K, the plane π of the four corresponding points M, M_1, M_2, M_{12} touches its envelope in the intersection Π of the lines MM_{12} and $M_1 M_2$; the parametric lines on the envelope form a conjugate system whose tangents are harmonic to the lines MM_{12} and $M_1 M_2$, and contain the focal points of the lines MM_1, MM_2, $M_{12} M_1$, $M_{12} M_2$ for the congruences generated by them.**

10. TRANSFORMATIONS T IN TANGENTIAL COÖRDINATES

When a surface S is referred to a conjugate system, if x, y, z, w and X, Y, Z, W are the point and tangential coördinates respectively of S so that

$$Xx + Yy + Zz + Ww = 0, \tag{80}$$

then X, Y, Z, W satisfy an equation of the form

$$\frac{\partial^2 \lambda}{\partial u \partial v} + \alpha \frac{\partial \lambda}{\partial u} + \beta \frac{\partial \lambda}{\partial v} + \gamma \lambda = 0. \tag{81}$$

Evidently the analytical theory of § 1 is independent of the geometrical interpretation there given, and has a meaning when applied to equation (81). This we will give and study the relation between the two sets of equations.

The adjoint of (81) is

$$\frac{\partial^2 \mu}{\partial u \partial v} - \alpha \frac{\partial \mu}{\partial u} - \beta \frac{\partial \mu}{\partial v} + \left(\gamma - \frac{\partial \alpha}{\partial u} - \frac{\partial \beta}{\partial v} \right) \mu = 0. \tag{82}$$

If λ_1 and μ_1 are solutions of these equations, the following integrals have a meaning:

$$\begin{aligned} \bar{\sigma}_1 &= \int \mu_1 \left(\frac{\partial \lambda_1}{\partial u} + \beta \lambda_1 \right) du + \lambda_1 \left(\frac{\partial \mu_1}{\partial v} - \alpha \mu_1 \right) dv, \\ \bar{\tau}_1 &= \int \lambda_1 \left(\frac{\partial \mu_1}{\partial u} - \beta \mu_1 \right) du + \mu_1 \left(\frac{\partial \lambda_1}{\partial v} + \alpha \lambda_1 \right) dv. \end{aligned} \tag{83}$$

The functions X_1, Y_1, Z_1, W_1 defined by equations of the form

$$\frac{\partial X_1}{\partial u} = \bar{\tau}_1 \frac{\partial}{\partial u}\left(\frac{X}{\lambda_1} \right), \qquad \frac{\partial X_1}{\partial v} = -\bar{\sigma}_1 \frac{\partial}{\partial v}\left(\frac{X}{\lambda_1} \right), \tag{84}$$

are the tangential coördinates of a second surface, upon which the parametric curves form a conjugate system.

* Cf. M_1, p. 409.

In addition to (80) we have the equations of condition

$$
\begin{aligned}
&X\frac{\partial x}{\partial u} + Y\frac{\partial y}{\partial u} + Z\frac{\partial z}{\partial u} + W\frac{\partial w}{\partial u} = 0, \\
(85) \qquad &X\frac{\partial x}{\partial v} + Y\frac{\partial y}{\partial v} + Z\frac{\partial z}{\partial v} + W\frac{\partial w}{\partial v} = 0.
\end{aligned}
$$

Consider the function

$$
(86) \qquad \theta_1 = X_1' x + Y_1' y + Z_1' z + W_1' w,
$$

where X_1', Y_1', Z_1', W_1' are given by integrals of the form

$$
(87) \qquad X_1' = \int \mu_1 \left(\frac{\partial X}{\partial u} + \beta X \right) du + X \left(\frac{\partial \mu_1}{\partial v} - \alpha \mu_1 \right) dv.
$$

In consequence of (81) and (85) we have

$$
(88) \qquad
\begin{gathered}
\frac{\partial \theta_1}{\partial u} = \sum \frac{\partial x}{\partial u} X_1' + \frac{\partial w}{\partial u} W_1', \qquad \frac{\partial \theta_1}{\partial v} = \sum \frac{\partial x}{\partial v} X_1' + \frac{\partial w}{\partial v} W_1', \\
\frac{\partial^2 \theta_1}{\partial u \partial v} = \sum \frac{\partial^2 x}{\partial u \partial v} X_1' + \frac{\partial^2 w}{\partial u \partial v} W_1',
\end{gathered}
$$

where as usual the symbol $\sum$ signifies the sum for three terms in x, y, z. Hence θ_1 is a solution of equation (1).*

In like manner it can be shown that λ_1 given by

$$
(89) \qquad \lambda_1 = X x_1' + Y y_1' + Z z_1' + W w_1',
$$

where x_1', y_1', z_1', w_1' are given by equations of the form (3), is a solution of (81). It is our purpose to show that equations (9) and (84) define the same transformation of S, when θ_1 and λ_1 are given by (86) and (89).

The analogue of equation (5) is

$$
(90) \qquad X_1 = X_1' - \frac{\bar{\sigma}_1}{\lambda_1} X.
$$

From (9) it follows that the points T_1 and T_2, whose coördinates ξ_1, η_1, ζ_1, ω_1; ξ_2, η_2, ζ_2, ω_2 are of the form

$$
(91) \qquad \xi_1 = \frac{\partial}{\partial v}\left(\frac{x}{\theta_1} \right), \qquad \xi_2 = \frac{\partial}{\partial u}\left(\frac{x}{\theta_1} \right),
$$

are the intersections of corresponding tangents to the parametric curves on S and S_1. Since

$$
(92) \qquad \frac{\partial \xi_2}{\partial v} = \frac{\partial \xi_1}{\partial u} = -\left(\frac{\partial \log \theta_1}{\partial v} + a \right) \xi_1 - \left(\frac{\partial \log \theta_1}{\partial u} + b \right) \xi_2,
$$

* Cf. Darboux, *Leçons*, vol. 2, p. 188.

the points T_1 and T_2 are the focal points of the congruence of lines $T_1 T_2$. These lines are the intersections of the tangent planes to S and S_1.

We shall show that X_1', Y_1', Z_1', W_1' are the tangential coördinates of the locus of T_1. In fact, it follows from (80), (85), (86), (87) and (88) that

$$
\sum \xi_1 X_1' + \omega_1 W_1' = 0, \tag{93}
$$
$$
\sum \xi_1 \frac{\partial X_1'}{\partial v} + \omega_1 \frac{\partial W_1'}{\partial v} = 0.
$$

Moreover, the condition

$$
\sum \xi_1 \frac{\partial X_1'}{\partial u} + \omega_1 \frac{\partial W_1'}{\partial u} = 0
$$

follows from

$$
\sum \frac{\partial x}{\partial v}\frac{\partial X}{\partial u} + \frac{\partial w}{\partial v}\frac{\partial W}{\partial u} = 0,
$$

which is a consequence of (85) and their derivatives.

In like manner it can be shown that the tangential coördinates X_1'', Y_1'', Z_1'', W_1'' of the locus of T_2 are of the form

$$
X_1'' = \int X\left(\frac{\partial \mu_1}{\partial u} - \beta\mu_1\right) du + \mu_1\left(\frac{\partial X}{\partial v} + \alpha X\right) dv. \tag{94}
$$

From (5), (86), (89) and (90) we have

$$
\theta_1 = \sum X_1 x + W_1 w, \tag{95}
$$
$$
\lambda_1 = \sum x_1 X + w_1 W.
$$

When the values of λ_1 and θ_1 from (89) and (86) are substituted in (4) and (83), the resulting equations are reducible to

$$
\sigma_1 = \int\left(\sum X_1' \frac{\partial x_1'}{\partial u} + W_1' \frac{\partial w_1'}{\partial u}\right) du + \left(\sum X_1' \frac{\partial x_1'}{\partial v} + W_1' \frac{\partial w_1'}{\partial v}\right) dv,
$$
$$
\bar{\sigma}_1 = \int\left(\sum x_1' \frac{\partial X_1'}{\partial u} + w_1' \frac{\partial W_1'}{\partial u}\right) du + \left(\sum x_1' \frac{\partial X_1'}{\partial v} + w_1' \frac{\partial W_1'}{\partial v}\right) dv.
$$

Hence by a suitable choice of the additive constants of integration we have

$$
\sigma_1 + \bar{\sigma}_1 = \sum X_1' x_1' + W_1' w_1'. \tag{96}
$$

In consequence of these results we have from (5) and (90)

$$
\sum X_1 x_1 + W_1 w_1 = 0.
$$

Since also

$$
\sum \xi_2 X_1' + \omega_2 W_1' = 0,
$$

it follows from (9), (90), (91) and (93) that

$$\sum X_1 \frac{\partial x_1}{\partial u} + W_1 \frac{\partial w_1}{\partial u} = 0, \qquad \sum X_1 \frac{\partial x_1}{\partial v} + W_1 \frac{\partial w_1}{\partial v} = 0.$$

Hence equations (9) and (84) define the same transformation T of S.

By making use of the results of § 5, we can obtain the equations of the theorem of permutability of transformations T from the standpoint of tangential coördinates. The functions λ_{12} and λ_{21} must satisfy

$$\frac{\partial \lambda_{ij}}{\partial u} = \bar{\tau}_i \frac{\partial}{\partial u}\left(\frac{\lambda_j}{\lambda_i}\right), \qquad \frac{\partial \lambda_{ij}}{\partial v} = -\bar{\sigma}_1 \frac{\partial}{\partial v}\left(\frac{\lambda_j}{\lambda_i}\right) \quad \left(\begin{matrix} i = 1, 2, \\ j = 1, 2, \end{matrix} \; i \neq j\right). \tag{97}$$

The functions $\bar{\sigma}_{12}$, $\bar{\sigma}_{21}$, $\bar{\tau}_{12}$, $\bar{\tau}_{21}$ are given by

$$\begin{aligned} \bar{\tau}_1 \bar{\tau}_{12} &= \bar{\tau}_2 \bar{\tau}_{21} = \frac{\lambda_2 \lambda_{21} \bar{\tau}_1}{\lambda_1} + \frac{\lambda_1 \lambda_{12} \bar{\tau}_2}{\lambda_2} - \lambda_{12} \lambda_{21}, \\ \bar{\sigma}_1 \bar{\sigma}_{12} &= \bar{\sigma}_2 \bar{\sigma}_{21} = -\left(\frac{\lambda_2 \lambda_{21} \bar{\sigma}_1}{\lambda_1} + \frac{\lambda_1 \lambda_{12} \bar{\sigma}_2}{\lambda_2} + \lambda_{12} \lambda_{21}\right), \end{aligned} \tag{98}$$

and the tangential coördinates of S_{12}, namely X_{12}, Y_{12}, Z_{12}, W_{12}, are of the form

$$\lambda_1 \lambda_{12} X_{12} = \lambda_2 \lambda_{21} X_1 + \lambda_1 \lambda_{12} X_2 - \lambda_{12} \lambda_{21} X. \tag{99}$$

If equations similar to (95) are to be satisfied, we must have

$$\begin{aligned} \lambda_{12} &= \sum X_1 x_{12} + W_1 w_{12} = \sum X_1 x_2 + W_1 w_2 - \theta_{21}, \\ \lambda_{21} &= \sum X_2 x_{12} + W_2 w_{12} = \sum X_2 x_1 + W_2 w_1 - \theta_{12}. \end{aligned} \tag{100}$$

When these equations are differentiated, we find that the resulting equations are satisfied in virtue of the preceding formulas. Hence we may take λ_{12} and λ_{21} as given by (100).

Equations similar to (5) and (90) are

$$x'_{12} = x_{12} + \frac{\sigma_{12}}{\theta_{12}} x_1 = x_2 - \frac{\theta_{21}}{\theta_1} x - \frac{x_1}{\sigma_1}\left(\frac{\theta_1 \sigma_2}{\theta_2} + \theta_{21}\right),$$

$$X'_{12} = X_{12} + \frac{\bar{\sigma}_{12}}{\lambda_{12}} X_1 = X_2 - \frac{\lambda_{21}}{\lambda_1} X - \frac{X_1}{\bar{\sigma}_1}\left(\frac{\lambda_1 \bar{\sigma}_2}{\lambda_2} + \lambda_{21}\right).$$

From these equations, (98) and (100) we obtain

$$\sum x'_{12} X'_{12} + w'_{12} W'_{12} = \sigma_{12} + \bar{\sigma}_{12}.$$

Consequently when λ_{12} and λ_{21} have the values (100), the expressions (99) are the tangential coördinates of S_{12} whose point coördinates are given by (45). From the form of (99) we are led at once to

THEOREM 8. *When S, S_1, S_2, S_{12} form a quatern for transformations T, four corresponding tangent planes meet in a point.*

During the remainder of this section we assume that the point coördinates are cartesian and that X, Y, Z are the direction-cosines of the normal to S. Consequently $-W$ is the distance from the origin to the tangent plane.

From (58) it follows that x_1' as it appears in (89) is equal to $x\sigma_1' + x^{(1)}$ and $w_1' = \sigma_1'$. Hence (89) may be replaced by

$$\lambda_1 = Xx^{(1)} + Yy^{(1)} + Zz^{(1)}. \tag{101}$$

Consequently λ_1 is the distance from the origin to the tangent plane to $S^{(1)}$.

We note that λ_1 and μ_1 determine a transformation of $S^{(1)}$. If X_1, Y_1, Z_1 are the direction-cosines of the normal to the transform $S_1^{(1)}$, and $-W_1^{(1)}$ the distance from the origin to the tangent plane to $S_1^{(1)}$, equations (84) are replaced by

$$\frac{\partial}{\partial u}\left(\frac{X_1}{W_1^{(1)}}\right) = -\bar{\tau}_1 \frac{\partial}{\partial u}\left(\frac{X}{\lambda_1}\right), \qquad \frac{\partial}{\partial v}\left(\frac{X_1}{W_1^{(1)}}\right) = \bar{\sigma}_1 \frac{\partial}{\partial v}\left(\frac{X}{\lambda_1}\right). \tag{102}$$

But X_1, Y_1, Z_1 are the direction-cosines of the normal to S_1 also. Moreover, the function W_1 is given by

$$\frac{\partial}{\partial u}\left(\frac{W_1}{W_1^{(1)}}\right) = -\bar{\tau}_1 \frac{\partial}{\partial u}\left(\frac{W}{\lambda_1}\right), \qquad \frac{\partial}{\partial v}\left(\frac{W_1}{W_1^{(1)}}\right) = \bar{\sigma}_1 \frac{\partial}{\partial v}\left(\frac{W}{\lambda_1}\right). \tag{103}$$

11. Transformations Ω of conjugate systems with equal tangential invariants

When equation (81) has equal invariants, it can be written

$$\frac{\partial^2 \lambda_1}{\partial u \partial v} + \frac{\partial \log \sqrt{\rho}}{\partial v}\frac{\partial \lambda_1}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u}\frac{\partial \lambda_1}{\partial v} + \gamma\lambda_1 = 0.$$

Analogously to the results of § 9 we have that $\mu_1 = 2\rho\lambda_1$ is a solution of the adjoint of this equation. For this value we have

$$\bar{\sigma}_1 = \tau_1 = \lambda_1^2 \rho,$$

so that the tangential equation of the transform is

$$\frac{\partial^2 \lambda_1'}{\partial u \partial v} - \frac{\partial}{\partial v}\log \sqrt{\rho}\lambda_1 \frac{\partial \lambda_1'}{\partial u} - \frac{\partial}{\partial u}\log \sqrt{\rho}\lambda_1 \frac{\partial \lambda_1'}{\partial v} + \gamma_1 \lambda_1' = 0,$$

which also has equal invariants.

If we put

$$\vartheta_1 = \lambda_1 \sqrt{\rho}, \qquad \vartheta_1' = \lambda_1' / \sqrt{\rho}\lambda_1,$$

these equations are equivalent to

$$\frac{\partial^2 \vartheta_1}{\partial u \partial v} = \left(\frac{1}{\sqrt{\rho}}\frac{\partial^2 \sqrt{\rho}}{\partial u \partial v} - \gamma\right)\vartheta_1, \qquad \frac{\partial^2 \vartheta_1'}{\partial u \partial v} = \left(\sqrt{\rho}\lambda_1 \frac{\partial^2}{\partial u \partial v}\frac{1}{\sqrt{\rho}\lambda_1} - \gamma_1\right)\vartheta_1'.$$

These equations are satisfied respectively by the functions

$$\nu_1 = \sqrt{\rho}X, \qquad \nu_2 = \sqrt{\rho}Y, \qquad \nu_3 = \sqrt{\rho}Z;$$

$$\bar{\nu}_1 = X_1/\sqrt{\rho}\lambda_1, \qquad \bar{\nu}_2 = Y_1/\sqrt{\rho}\lambda_1, \qquad \bar{\nu}_3 = Z_1/\sqrt{\rho}\lambda_1.$$

In terms of these functions equations (84) are reducible to

$$(84') \qquad \begin{aligned} \frac{\partial}{\partial u}(\bar{\nu}_i\,\vartheta_1) &= \left(\vartheta_1\frac{\partial \nu_i}{\partial u} - \nu_i\frac{\partial \vartheta_1}{\partial u}\right), \\ \frac{\partial}{\partial v}(\bar{\nu}_i\,\vartheta_1) &= -\left(\vartheta_1\frac{\partial \nu_i}{\partial v} - \nu_i\frac{\partial \vartheta_1}{\partial v}\right) \end{aligned} \qquad (i = 1, 2, 3).$$

Since the tangential equation of S has equal invariants, there exists a surface Σ with this spherical representation of its asymptotic lines. Its point coördinates ξ, η, ζ are given by the Lelieuvre formulas

$$\frac{\partial \xi}{\partial u} = \nu_2\frac{\partial \nu_3}{\partial u} - \nu_3\frac{\partial \nu_2}{\partial u}, \qquad \frac{\partial \xi}{\partial v} = -\nu_2\frac{\partial \nu_3}{\partial v} + \nu_3\frac{\partial \nu_2}{\partial v}.*$$

Similar equations in the functions ν_i give the point coördinates of a surface Σ_1 with the same spherical representation of its asymptotic lines as the parametric system on S_1. Moreover, equations (84′) are the condition that Σ and Σ_1 be the focal surfaces of a W-congruence.†

From the theory of W-congruences it follows that, if X, Y, Z and X_1, Y_1, Z_1 are the direction-cosines (not merely direction-parameters), of the normals to Σ and Σ_1 respectively, then

$$\nu_1 = \sqrt{\rho}X, \qquad \bar{\nu}_1 = \sqrt{\rho_1}\,X_1,$$

where $-1/\rho^2$ and $-1/\rho_1^2$ are the gaussian curvatures of Σ and Σ_1 respectively. Hence (84′) may be written

$$\frac{\partial}{\partial u}(\sqrt{\rho\rho_1}\,\lambda_1 X_1) = \rho\lambda_1^2\frac{\partial}{\partial u}\left(\frac{X}{\lambda_1}\right), \qquad \frac{\partial}{\partial v}(\sqrt{\rho\rho_1}\,\lambda_1 X_1) = -\rho\lambda_1^2\frac{\partial}{\partial v}\left(\frac{X}{\lambda_1}\right).$$

Comparing these equations with (102), we have

$$W_1^{(1)} = \frac{1}{\sqrt{\rho\rho_1}\,\lambda_1},$$

and equations (103) become

$$\frac{\partial}{\partial u}(\sqrt{\rho\rho_1}\,\lambda_1 W_1) = -\rho\lambda_1^2\frac{\partial}{\partial u}\left(\frac{W}{\lambda_1}\right), \qquad \frac{\partial}{\partial v}(\sqrt{\rho\rho_1}\,\lambda_1 W_1) = \rho\lambda_1^2\frac{\partial}{\partial v}\left(\frac{W}{\lambda_1}\right).$$

But these are the equations of transformations Ω of conjugate systems with equal tangential invariants previously found by the author.‡ Moreover, the

* E., p. 193. A reference of this sort is to the author's *Differential Geometry.*

† E., p. 419.

‡ M_2, §§ 1, 3.

special surfaces $S^{(1)}$ and $S_1^{(1)}$, treated in § 7, were used in the discussion of the transformations Ω.*

12. The extended theorem of permutability

In this section we extend the theorem of permutability so as to involve a group of eight surfaces. Let S_1, S_2, S_3 be three transforms of S by means of functions θ_i, ϕ_i for $i = 1, 2, 3$ respectively. Applying the theorem of permutability to the three pairs of these surfaces, we get three new surfaces S_{12}, S_{23}, S_{31}. We recall that $S_{ij} = S_{ji}$. Since S_{12} and S_{13} are transforms of S_1, there exists a family of surfaces S', for each of which S_1, S_{12}, S_{13}, S' form a quatern. It is our purpose to show that one of these surfaces S' is such that S_2, S_{12}, S_{23}, S' form a quatern; and likewise S_3, S_{13}, S_{23}, S'.

We denote by θ'_{12}, w'_{12}, ϕ'_{12} the functions transforming S_{12} into S'. The equations analogous to the first of (35) and (36) are

$$\begin{aligned} \theta_{12}\,\theta'_{12}\,w'_{12} &= w_{13}(\theta_{13}\,\theta'_{13} + \theta_{12}\,\theta'_{12} - \theta'_{12}\,\theta'_{13}), \\ \theta_{12}\,\theta'_{12}\,w'_{12}\,x' &= w_{13}(\theta_{13}\,\theta'_{13}\,x_{12} + \theta_{12}\,\theta'_{12}\,x_{13} - \theta'_{13}\,\theta'_{12}\,x_1). \end{aligned} \tag{104}$$

In consequence of (36) and an analogous expression for x_{13} the second of (104) is reducible to

$$\begin{aligned} \theta_{12}\,\theta'_{12}\,w'_{12}\,x' = x_1 \Bigg(& w_{12}\,\theta'_{12}\,w_{13}\,\theta'_{13} \\ & - \frac{\theta_2\,\theta_{21}}{\theta_1\,\theta_{12}} w_{13}\,\theta'_{13}\,w_2\,\theta_{13} - \frac{\theta_3\,\theta_{31}}{\theta_1\,\theta_{13}} w_{12}\,\theta'_{12}\,w_3\,\theta_{12} \Bigg) \\ & - x_2\,w_{13}\,\theta'_{13}\,w_2\,\theta_{13} - x_3\,w_{12}\,\theta'_{12}\,w_3\,\theta_{12} \\ & + \frac{x}{\theta_1}(w_2\,\theta_{21}\,w_{13}\,\theta'_{13}\,\theta_{13} + w_3\,\theta_{31}\,w_{12}\,\theta'_{12}\,\theta_{12}). \end{aligned} \tag{105}$$

In deriving these equations, we looked upon S' as a transform of S_{12} which in turn is a transform of S_1. Looking upon S_{12} as a transform of S_2 we get the analogous equations

$$\begin{aligned} \theta_{21}\,\theta'_{21}\,w'_{21} &= w_{23}(\theta_{21}\,\theta'_{21} + \theta_{23}\,\theta'_{23} - \theta'_{21}\,\theta'_{23}), \\ \theta_{21}\,\theta'_{21}\,w'_{21}\,x' &= w_{23}(\theta_{21}\,\theta'_{21}\,x_{12} + \theta_{23}\,\theta'_{23}\,x_{23} - \theta'_{21}\,\theta'_{23}\,x_2). \end{aligned}$$

The latter equation is reducible to

$$\begin{aligned} \theta_{21}\,\theta'_{21}\,w'_{21}\,x' = & - x_1\,w_{23}\,\theta'_{23}\,w_1\,\theta_{23} - x_3\,w_{21}\,\theta'_{21}\,w_3\,\theta_{21} \\ & + x_2 \Bigg(w_{21}\,\theta'_{21}\,w_{23}\,\theta'_{23} - w_{21}\,\theta'_{21}\,w_3\,\theta_{21}\frac{\theta_3\,\theta_{32}}{\theta_2\,\theta_{23}} \\ & \qquad - w_{23}\,\theta'_{23}\,w_1\,\theta_{23}\frac{\theta_1\,\theta_{12}}{\theta_2\,\theta_{21}} \Bigg) \\ & + \frac{x}{\theta_2}(w_3\,\theta_{32}\,w_{21}\,\theta'_{21}\,\theta_{21} + w_1\,\theta_{12}\,w_{23}\,\theta'_{23}\,\theta_{23}). \end{aligned} \tag{106}$$

* M_2, § 6.

From their definition it follows that θ'_{21} and w'_{21} are the same functions as θ'_{12} and w'_{12} respectively. Making use of this fact, we eliminate x' from (105) and (106). In the reduction we note that from (35) we have

$$w_1 w_{12} \theta_1 \theta_{12} = w_2 w_{21} \theta_2 \theta_{21}.$$

The resulting equation is of the form

$$Ax_1 + Bx_2 + Cx = 0,$$

where A, B, and C are determinate functions. These functions must equal zero, since equations similar to the above hold also in the y's and z's. This gives the three equations

$$\begin{aligned}
&\theta_1 w_{12} \theta'_{12} w_{13} \theta'_{13} - \theta_2 w_{13} \theta'_{13} w_2 \theta_{21} \frac{\theta_{13}}{\theta_{12}} - \theta_3 w_{12} \theta'_{12} w_3 \theta_{31} \frac{\theta_{12}}{\theta_{13}} \\
&\qquad\qquad + \theta_2 w_2 \theta_{23} w_{23} \theta'_{23} = 0, \\
&\theta_2 w_{21} \theta'_{21} w_{23} \theta'_{23} - \theta_3 w_{21} \theta'_{21} w_3 \theta_{32} \frac{\theta_{21}}{\theta_{23}} - \theta_1 w_{23} \theta'_{23} w_1 \theta_{12} \frac{\theta_{23}}{\theta_{21}} \\
&\qquad\qquad + \theta_1 w_1 \theta_{13} w_{13} \theta'_{13} = 0, \\
&w_{13} \theta'_{13} w_1 \theta_{13} w_2 \theta_{21} + w_{12} \theta'_{12} w_1 \theta_{12} w_3 \theta_{31} - w_{21} \theta'_{21} w_2 \theta_{21} w_3 \theta_{32} \\
&\qquad\qquad - w_{23} \theta'_{23} w_1 \theta_{12} w_2 \theta_{23} = 0.
\end{aligned} \tag{107}$$

It is readily found that these equations are equivalent to the three

$$\theta_i w_{ij} \theta'_{ij} = \theta_j w_j \theta_{ji} \frac{\theta_{ik}}{\theta_{ij}} + \theta_i w_j \theta_{jk} - \theta_k w_j \theta_{ji} \qquad \left(\begin{matrix} i = 1, 2, \; j = 1, 2, \; k = 1, 2 \\ i \neq j \neq k \end{matrix} \right). \tag{108}$$

When these values are substituted in (105), the result is reducible to

$$\begin{aligned}
\Phi x' = {} & x_1 (\theta_2 \theta_{31} \theta_{23} + \theta_3 \theta_{21} \theta_{32} - \theta_1 \theta_{23} \theta_{32}) \\
& + x_2 (\theta_1 \theta_{13} \theta_{32} + \theta_3 \theta_{12} \theta_{31} - \theta_2 \theta_{31} \theta_{13}) \\
& + x_3 (\theta_1 \theta_{23} \theta_{12} + \theta_2 \theta_{21} \theta_{13} - \theta_3 \theta_{21} \theta_{12}) \\
& + x (\theta_{12} \theta_{23} \theta_{31} + \theta_{21} \theta_{13} \theta_{32}),
\end{aligned} \tag{109}$$

where

$$\begin{aligned}
\Phi &= \frac{\theta_1 \theta_{12} w_{12} \theta'_{12} w'_{12}}{w_2 w_3} \\
&= \theta_1 (\theta_{13} \theta_{32} + \theta_{12} \theta_{23} - \theta_{32} \theta_{23}) + \theta_2 (\theta_{31} \theta_{23} + \theta_{21} \theta_{13} - \theta_{13} \theta_{31}) \\
&\quad + \theta_3 (\theta_{12} \theta_{31} + \theta_{32} \theta_{21} - \theta_{12} \theta_{21}) + \theta_{13} \theta_{32} \theta_{21} + \theta_{12} \theta_{23} \theta_{31}.
\end{aligned} \tag{110}$$

When we proceed with S' as a transform of S_{13} or S_{23}, we arrive at the same result, which evidently is of symmetric form.

It remains for us to show that the functions θ'_{ij} as given by (108) satisfy equations analogous to (26), namely

$$(111)\qquad \frac{\partial}{\partial u}(w_{ij}\,\theta'_{ij}) = \tau_{ij}\frac{\partial}{\partial u}\left(\frac{\theta_{ik}}{\theta_{ij}}\right), \qquad \frac{\partial}{\partial v}(w_{ij}\,\theta'_{ij}) = -\,\sigma_{ij}\frac{\partial}{\partial v}\left(\frac{\theta_{ik}}{\theta_{ij}}\right).$$

We know that this is true, since (108) for $i = 1$, $j = 2$, $k = 3$ follows from (36) when x_{12}, x_1, x_2, x are replaced by θ'_{12}, θ_{13}, θ_{23}, θ_3 respectively; and these results are general. Hence we have the extended theorem of permutability:

THEOREM 9. *If S, S_1, S_2, S_{12}; S, S_2, S_3, S_{23}; S, S_3, S_1, S_{13} are three quaterns of surfaces, a surface S' can be found, without quadrature, such that S_1, S_{12}, S_{13}, S'; S_2, S_{12}, S_{23}, S'; S_3, S_{13}, S_{23}, S' are quaterns.*

13. Relations between transformations T and radial transformations

If ω is a solution of equation (18), the surface $\bar{S}$ whose coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$, are given by

$$(112)\qquad \bar{x} = \frac{x}{\omega}, \qquad \bar{y} = \frac{y}{\omega}, \qquad \bar{z} = \frac{z}{\omega},$$

is referred to a conjugate system. In fact, the point equation of $\bar{S}$ is

$$(113)\qquad \frac{\partial^2 \bar{\theta}}{\partial u \partial v} + \left(a + \frac{\partial \log \omega}{\partial v}\right)\frac{\partial \bar{\theta}}{\partial u} + \left(b + \frac{\partial \log \omega}{\partial u}\right)\frac{\partial \bar{\theta}}{\partial v} = 0.$$

We say that $\bar{S}$ is obtained from S by a *radial* transformation, since the line joining any pair of corresponding points on S and $\bar{S}$ passes through a point—the origin.

If θ_1 is a solution of (18), then $\bar{\theta}_1 = \theta_1/\omega$ is a solution of (113). Also it can be shown that if ϕ_1 is a solution of the adjoint equation of (18), then $\bar{\phi}_1 = \phi_1\,\omega$ is a solution of the adjoint of (113).

We consider the transformation T of $\bar{S}$ by means of these functions $\bar{\theta}_1$ and $\bar{\phi}_1$. If $\bar{\tau}_1$ and $\bar{\sigma}_1$ denote functions analogous to τ_1 and σ_1, it is readily found that to within additive constants we have

$$\bar{\tau}_1 = \tau_1, \qquad \bar{\sigma}_1 = \sigma_1.$$

Assuming these values, we have from equations (20) and the analogous ones,

$$\frac{\partial}{\partial u}(\bar{x}_1\,\bar{w}_1) = \bar{\tau}_1\frac{\partial}{\partial u}\left(\frac{\bar{x}}{\bar{\theta}_1}\right), \qquad \frac{\partial}{\partial v}(\bar{x}_1\,\bar{w}_1) = -\,\bar{\sigma}_1\frac{\partial}{\partial v}\left(\frac{\bar{x}}{\bar{\theta}_1}\right),$$

by integration

$$(114)\qquad \bar{x}_1\,\bar{w}_1 = x_1\,w_1$$

to within an additive constant. Also $\bar{w}_1$ is given by

$$(115)\qquad \frac{\partial \bar{w}_1}{\partial u} = \bar{\tau}_1\frac{\partial}{\partial u}\left(\frac{1}{\bar{\theta}_1}\right) = \tau_1\frac{\partial}{\partial u}\left(\frac{\omega}{\theta_1}\right), \qquad \frac{\partial \bar{w}_1}{\partial v} = -\,\sigma_1\frac{\partial}{\partial v}\left(\frac{\omega}{\theta_1}\right).$$

Evidently there exists a function ω_1 defined by

$$(116) \qquad \frac{\partial}{\partial u}(\omega_1 w_1) = \tau_1 \frac{\partial}{\partial u}\left(\frac{\omega}{\theta_1}\right), \qquad \frac{\partial}{\partial v}(\omega_1 w_1) = -\sigma_1 \frac{\partial}{\partial v}\left(\frac{\omega}{\theta_1}\right),$$

which equations are similar to (20). Comparing (115) and (116), we note that ω_1 can be chosen so that

$$(117) \qquad \omega_1 w_1 = \bar{w}_1,$$

and consequently we have from (114)

$$(118) \qquad \bar{x}_1 = \frac{x_1}{\omega_1}, \qquad \bar{y}_1 = \frac{y_1}{\omega_1}, \qquad \bar{z}_1 = \frac{z_1}{\omega_1}.$$

Hence we have

THEOREM 10. *If two surfaces S and $\bar{S}$ are in the relation of a radial transformation and S_1 is a T transform of S, a surface $\bar{S}_1$ can be found by a quadrature which is a radial transform of S_1 and a T transform of $\bar{S}$.*

When in particular $\omega = \theta_1$, then $\bar{\theta}_1 = 1$, and consequently $\bar{S}$ and $\bar{S}_1$ are parallel. Now in all generality we take

$$\bar{w}_1 = -1, \qquad \omega_1 = -1/w_1.$$

Therefore, we have

THEOREM 11. *A transformation T is equivalent to the combination of an axial, a parallel and an axial transformations.*

Consider now a general quatern of surfaces S, S_1, S_2, S_{12}. From (116), (26) and analogous equations it follows that the functions

$$\omega = \theta_2, \qquad \omega_1 = \theta_{12}, \qquad \omega_2 = \frac{1}{w_2},$$

determine axial transformations of S, S_1 and S_2 respectively, into $\bar{S}$, $\bar{S}_1$, $\bar{S}_2$. The equations determining the axial transform of S_{12} as of the pair S_1 and S_{12}, are

$$\frac{\partial}{\partial u}(\omega_{12} w_{12}) = \tau_{12} \frac{\partial}{\partial u}\left(\frac{\omega_1}{\theta_{12}}\right), \qquad \frac{\partial}{\partial v}(\omega_{12} w_{12}) = -\sigma_{12} \frac{\partial}{\partial v}\left(\frac{\omega_1}{\theta_{12}}\right).$$

In consequence of the preceding equations we may take $\omega_{12} = 1/w_{12}$.

In order to show that the same function ω_{12} determines the axial transform of S_{12}, as of the pair S_2 and S_{12}, the following equations must be satisfied:

$$(119) \qquad \frac{\partial}{\partial u}\left(\frac{w_{21}}{w_{12}}\right) = \tau_{21} \frac{\partial}{\partial u}\left(\frac{1}{w_2 \theta_{21}}\right), \qquad \frac{\partial}{\partial v}\left(\frac{w_{21}}{w_{12}}\right) = -\sigma_{21} \frac{\partial}{\partial v}\left(\frac{1}{w_2 \theta_{21}}\right).$$

When the values of w_{12}, w_{21}, σ_{21} and τ_{21}, as given by (35) and analogous equations are substituted, it is found that (119) are satisfied. Hence the surfaces

$\bar{S}$, $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_{12}$ form a quatern. Their coördinates are of the form

$$\bar{x} = \frac{x}{\theta_2}, \qquad \bar{x}_1 = \frac{x_1}{\theta_{12}}, \qquad \bar{x}_2 = x_2\, w_2, \qquad \bar{x}_{12} = x_{12}\, w_{12},$$

the transformation functions being

$$\begin{aligned}
\bar{\theta}_1 &= \frac{\theta_1}{\theta_2}, & \bar{\phi}_1 &= \theta_2\,\phi_1, & \bar{w}_1 &= w_1\,\theta_{12};\\
\bar{\theta}_2 &= 1, & \bar{\phi}_2 &= \theta_2\,\phi_2, & \bar{w}_2 &= 1;\\
\bar{\theta}_{12} &= 1, & \bar{\phi}_{12} &= \theta_{12}\,\phi_{12}, & \bar{w}_{12} &= 1;\\
\bar{\theta}_{21} &= w_2\,\theta_{21}, & \bar{\phi}_{21} &= \frac{\phi_{21}}{w_2}, & \bar{w}_{21} &= \frac{w_{21}}{w_{12}}.
\end{aligned}$$

These functions satisfy equations analogous to (35).

In a similar manner we get a second quatern by using θ_1 for the axial transformation of S. Hence we have

THEOREM 12. *When a quatern of surfaces is known, two other quaterns each containing two pairs of parallel surfaces can be found without quadrature, and these surfaces are axial transforms of the surfaces of the given quatern.*

As a matter of fact axial transformations can be looked upon as special types of transformations T. For if we take

$$\tau_2 = -\sigma_2 = 1, \qquad \theta_2 = \omega - 1, \qquad w_2 = \frac{1}{\theta_2} + 1 = \frac{\omega}{\omega - 1},$$

equations of the form (19) and (20) with subscripts 2 instead of 1 are integrable in the form

$$x_2 = \frac{x}{\omega}, \qquad y_2 = \frac{y}{\omega}, \qquad z_2 = \frac{z}{\omega}.$$

Let us apply the theorem of permutability to the case in which S_2 is given as above. One solution of (37) is

$$\theta_{21} = \frac{\theta_1}{\omega_2\,\theta_2} = \frac{\theta_1}{\omega}.$$

We find also

$$\tau_{12} = -\sigma_{12} = 1, \qquad \sigma_{21} = \sigma_1, \qquad \tau_{21} = \tau_1,$$

$$w_{12}\,\theta_{12} = 1 + \theta_{12}, \qquad w_{21} = w_1(1 + \theta_{12}).$$

Hence if we put $\omega_1 = \theta_{12} + 1$, equation (36) in this case reduces to (118). Consequently the theorem of permutability is equally true when an axial transformation is used. It is readily shown also that theorem 12 can be established by means of the generalized results of § 11.

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